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Biharmonic maps between warped product manifolds

A. Balmuş[a], S. Montaldo[a,*], C. Oniciuc[b]

[a] Università degli Studi di Cagliari, Dipartimento di Matematica, Via Ospedale 72, 09124 Cagliari, Italy
[b] Faculty of Mathematics, "Al.I. Cuza" University of Iasi, Bd. Carol I Nr. 11, 700506 Iasi, Romania

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Abstract

Biharmonic maps between warped products are studied. The main results are:

(i) the condition for the biharmonicity of the inclusion of a Riemannian manifold N into the warped product $M \times_{f^2} N$ and of the projection $\overline{\pi} : M \times_{f^2} N \to M$;
(ii) the construction of two new classes of non-harmonic biharmonic maps using products of harmonic maps $\phi = \mathbf{1}_M \times \psi : M \times N \to M \times N$ and warping the metric on their domain or codomain;
(iii) the study of three classes of axially symmetric biharmonic maps, using the warped product setting.

MSC: 58E20

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1. Introduction

Harmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds are the critical points of the energy $E(\phi) = \frac{1}{2}\int_M |\mathrm{d}\phi|^2 v_g$, and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field $\tau(\phi) = \mathrm{trace}\nabla \mathrm{d}\phi$. As suggested by Eells and Sampson in [13], we can define the *bienergy* of a map ϕ by $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$, and say that ϕ is *biharmonic* if it is a critical point of the bienergy.

In [19,20] Jiang derived the first and the second variation formula for the bienergy, showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(\phi) &= -J^{\phi}(\tau(\phi)) = -\Delta\tau(\phi) - \mathrm{trace}\, R^N(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi \\ &= 0,\end{aligned}$$

where J^{ϕ} is the Jacobi operator of ϕ. The equation $\tau_2(\phi) = 0$ is called the *biharmonic equation*. Since J^{ϕ} is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

* Corresponding author.
E-mail addresses: balmus@unica.it (A. Balmuş), montaldo@unica.it (S. Montaldo), oniciucc@uaic.ro (C. Oniciuc).

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Biharmonic maps have been extensively studied in the last decade.

In [9] the authors completely classified the biharmonic submanifolds of the three-dimensional sphere, while in [10] there were given new methods to construct biharmonic submanifolds of codimension greater than one in the n-dimensional sphere. The biharmonic submanifolds into a space of nonconstant sectional curvature were also investigated. The proper biharmonic curves on Riemannian surfaces were studied in [11]. In [12] the authors obtained the parametric equations of all proper biharmonic curves of the Heisenberg group, and Inoguchi classified the biharmonic Legendre curves and the Hopf cylinders in three-dimensional Sasakian space forms (see [18]). Then, Sasahara gave in [30] the explicit representation of the proper biharmonic Legendre surfaces in five-dimensional Sasakian space forms.

The second variation formula for biharmonic maps in spheres was deduced [26] and the stability of certain classes of biharmonic maps in spheres was discussed in [23,24]. Also, in [31] there were given some sufficient conditions for the instability of Legendre proper biharmonic submanifolds in Sasakian space forms and the author proved the instability of Legendre curves and surfaces in Sasakian space forms.

In [2] the authors found new examples of biharmonic maps by conformally deforming the domain metric of harmonic ones. In this vein, new examples of biharmonic maps between the n-dimensional Euclidean sphere and the $(n + 1)$-dimensional sphere endowed with a special metric, conformally equivalent to the canonical one, were constructed in [27], while in [4] the author analyzed the behavior of the biharmonic equation under the conformal change of metric on the target manifold of harmonic Riemannian submersions. Moreover, in [28] the author gave some extensions of the results in [4] together with some further constructions of biharmonic maps.

In this note, we extend the idea of studying the effect of conformal changes of metric to the biharmonic equation by considering the situation of warped products. The concept of warped product plays an important role in differential geometry as well as in theoretical physics. The notion of warped metric was first introduced by Bishop and O'Neill in [8]. Since then, in Riemannian geometry, warped products have offered new examples of Riemannian manifolds with special curvature properties like Einstein spaces (see [7,22]), or (locally) symmetric spaces (see [5]). In Lorentzian geometry some important concepts can be expressed in terms of warped products and the geometric properties of Lorentzian warped products are of great interest due to their relativistic applications (see [6,7]).

The present article is organized as follows.

We first study the biharmonicity of the inclusion of a Riemannian manifold N into the warped product $M \times_{f^2} N$. With this setting we obtain new examples of proper biharmonic maps and rediscover some of the examples first obtained in [9] and [11]. In Section 4 we consider the product of two harmonic maps $\phi = \mathbf{1}_M \times \psi : M \times N \to M \times N$. By warping the metric on the domain or codomain we lose the harmonicity; nevertheless, under certain conditions on the warping function the product map remains biharmonic. In the case the product map is the identity map $\bar{\mathbf{1}} : M \times_{f^2} N \to M \times N$ we shall call the warping function, which is a solution of the above problem, a *biharmonic warping function*. In the instance that M is an Einstein manifold we show that the isoparametric functions provide examples of biharmonic warping ones. We also give the complete classification of the biharmonic warping functions when $M = \mathbb{R}$.

In the last section we use the warped product setting to study axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to a n-dimensional space form. We discuss certain examples and, when the target manifold is $\mathbb{R}^n \setminus \{0\}$, we give the complete classification of biharmonic axially symmetric maps. From the former classification we also deduce that the generalised Kelvin transformation $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$, $\phi(y) = y/|y|^\ell$ is a proper biharmonic map if and only if $m = \ell + 2$.

In some cases the classification of biharmonic maps leads to ordinary differential equations. The solutions of these ODE's are found with the aid of Mathematica and are amenable to techniques described in [21].

2. Preliminary

2.1. Biharmonic maps between Riemannian manifolds

Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. The tension field of ϕ is given by $\tau(\phi) = \operatorname{trace} \nabla d\phi$, and, for any compact domain $\Omega \subseteq M$, the *bienergy* is defined by

$$E_2(\phi) = \frac{1}{2} \int_\Omega |\tau(\phi)|^2 v_g.$$

Then we call *biharmonic* a smooth map ϕ which is a critical point of the bienergy functional for any compact domain $\Omega \subseteq M$. The first variation formula for the bienergy functional is given by

$$\left.\frac{\mathrm{d}E_2(\phi_t)}{\mathrm{d}t}\right|_{t=0} = \int_\Omega \langle \tau_2(\phi), V\rangle v_g,$$

where v_g is the volume element, while V is the variational vector field associated to the variation $\{\phi_t\}$ of ϕ, and

$$\tau_2(\phi) = -\Delta\tau(\phi) - \operatorname{trace} R^N\,(\mathrm{d}\phi, \tau(\phi))\,\mathrm{d}\phi. \tag{2.1}$$

Here Δ denotes the rough Laplacian on sections of the pull-back bundle $\phi^{-1}(TN)$ which is defined, with respect to a local orthonormal frame field $\{E_i\}$ on M, as

$$\Delta V = -\sum_{i=1}^{m}\left\{\nabla_{E_i}\nabla_{E_i}V - \nabla_{\nabla_{E_i}E_i}V\right\}, \quad V \in C\left(\phi^{-1}(TN)\right), \tag{2.2}$$

where ∇ is the induced connection in the pullback bundle $\phi^{-1}(TN)$.

Here are some remarks on biharmonic maps.

(A) A map ϕ is biharmonic if and only if its tension field is in the kernel of the Jacobi operator;
(B) a harmonic map is obviously a biharmonic map;
(C) a harmonic map is an absolute minimum of the bienergy;
(D) if M is compact and $\mathrm{Riem}^N \leq 0$, then $\phi : M \to N$ is biharmonic if and only if it is harmonic;
(E) if $\phi : M \to N$ is a Riemannian immersion with $|\tau(\phi)| = $ constant and $\mathrm{Riem}^N \leq 0$, then ϕ is biharmonic if and only if it is harmonic.

The first three remarks are immediate consequences of the definition of the bienergy and of Eq. (2.1). Properties **(D)** and **(E)** are proved in [19] and in [25], respectively.

2.2. *Riemannian structure of warped products*

Let M and N be two Riemannian manifolds equipped with Riemannian metrics g and h, respectively, and let $f \in C^\infty(M)$ be a positive function. Consider the product manifold $M \times N$ and denote by $\pi : M \times N \to M$ and $\eta : M \times N \to N$ its projections. The *warped product* $M \times_{f^2} N$ is the product manifold $M \times N$ endowed with the Riemannian metric G_{f^2} defined, for $X, Y \in T_{(x,y)}(M \times N)$, by

$$G_{f^2}(X, Y) = g(\mathrm{d}\pi(X), \mathrm{d}\pi(Y)) + (f \circ \pi)^2 h(\mathrm{d}\eta(X), \mathrm{d}\eta(Y)).$$

The function f is called the *warping function* of the warped product.

Let $X, Y \in C(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, where $X_1, Y_1 \in C(TM)$ and $X_2, Y_2 \in C(TN)$. Denote by ∇ the Levi-Civita connection on the Riemannian product $M \times N$ with respect to the product metric G and by R its curvature tensor field. The Levi-Civita connection $\widetilde{\nabla}$ of $M \times_{f^2} N$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2f^2}X_1(f^2)(0, Y_2) + \frac{1}{2f^2}Y_1(f^2)(0, X_2) - \frac{1}{2}h(X_2, Y_2)(\operatorname{grad} f^2, 0), \tag{2.3}$$

and the relation between the curvature tensor fields of G_{f^2} and G is

$$\begin{aligned}\widetilde{R}(X, Y) - R(X, Y) = \frac{1}{2f^2}\Bigg\{&\left(\nabla^M_{Y_1}\operatorname{grad} f^2 - \frac{1}{2f^2}Y_1(f^2)\operatorname{grad} f^2, 0\right) \wedge_{G_{f^2}} (0, X_2)\\ &- \left(\nabla^M_{X_1}\operatorname{grad} f^2 - \frac{1}{2f^2}X_1(f^2)\operatorname{grad} f^2, 0\right) \wedge_{G_{f^2}} (0, Y_2)\\ &- \frac{1}{2f^2}|\operatorname{grad} f^2|^2 (0, X_2) \wedge_{G_{f^2}} (0, Y_2)\Bigg\},\end{aligned} \tag{2.4}$$

where

$$(X \wedge_{G_{f^2}} Y)Z = G_{f^2}(Z, Y)X - G_{f^2}(Z, X)Y,$$

for all $X, Y, Z \in C(T(M \times N))$, (see, for example, [5]).

2.3. *Affine functions*

Following, for example, [17,29], a function $\alpha \in C^\infty(M)$ on a Riemannian manifold (M, g) is called an *affine function* if $\alpha \circ \gamma : I \subset \mathbb{R} \to \mathbb{R}$ is an affine function for any geodesic γ on M. If we denote by Hess $\alpha(X, Y) = X(Y\alpha) - (\nabla_X Y)\alpha$, $X, Y \in C(TM)$, the Hessian of a function $\alpha \in C^\infty(M)$, we have the following characterization.

Lemma 2.1 (*[29]*). *Let (M, g) be a Riemannian manifold and $\alpha \in C^\infty(M)$. Then the following statements are equivalent:*

(1) *α is an affine function;*
(2) grad α *is a parallel vector field;*
(3) *the Hessian of α vanishes identically;*
(4) grad α *is a Killing vector field.*

Remark 2.2. From the characterization of affine functions it follows that they are totally geodesic and thus harmonic. In particular, when M is compact any affine function is constant.

The classification result for complete, simply connected manifolds admitting a non-trivial affine function was given in [17], and then treated as a particular case in [29]. We present here another proof of this classification, based on a result of Svensson in [32].

Proposition 2.3. (a) *Let c be a positive real constant and take (P, q) to be a Riemannian manifold. Then the projection map $\overline{\eta} : P \times_{c^2} \mathbb{R} \to \mathbb{R}$ is an affine function on the warped product $P \times_{c^2} \mathbb{R}$.*
(b) *Any complete, simply connected Riemannian manifold (M, g) admitting a non-trivial affine function is isometric to a warped product $P \times_{c^2} \mathbb{R}$, with $c \in \mathbb{R}_+$. Moreover, the affine function is in this case the projection map $\overline{\eta} : P \times_{c^2} \mathbb{R} \to \mathbb{R}$.*

Proof. In order to prove (a), consider $\{\partial/\partial t\}$ to be the orthonormal basis on $\mathbb{R}$ with the Euclidean metric. Then grad $\overline{\eta} = \frac{1}{c^2}\partial/\partial t$ and this obviously is a parallel vector field on $P \times_{c^2} \mathbb{R}$.

To prove (b), we first use the fact that any horizontally homothetic submersion $\phi : (M, g) \to (N, h)$ with totally geodesic fibres and integrable horizontal distribution is locally the projection of a warped product onto its second factor [32]. In addition to this, if (M, g) is complete and M and N are simply connected, it is globally such a projection.

In the following we will prove that any non-trivial affine function is a horizontally homothetic submersion with totally geodesic fibres and integrable horizontal distribution.

Denote by α the non-trivial affine function on M. As $|\text{grad } \alpha| = \text{const} > 0$, α is a submersion with vertical distribution

$$T^V(M) = \ker \mathrm{d}\alpha = (\text{grad } \alpha)^\perp,$$

and integrable horizontal distribution

$$T^H(M) = \text{span}\{\text{grad } \alpha\}.$$

For $v, w \in T_x^H M$, with $v = c_1 \text{grad } \alpha$ and $w = c_2 \text{grad } \alpha$, we have

$$\begin{aligned} h_{\alpha(x)}(\mathrm{d}\alpha(v), \mathrm{d}\alpha(w)) &= c_1 c_2 |\text{grad } \alpha|_x^4 \\ &= |\text{grad } \alpha|_x^2 g(v, w), \end{aligned}$$

and α is a horizontally conformal submersion of constant dilation, and so a horizontally homothetic submersion.

As α is an affine function, $\nabla_X \text{grad } \alpha = 0$ and the second fundamental form of any fibre is identically zero. So α has totally geodesic fibres.

By summing all of the above we conclude that M is isometric to a warped product $P \times_{f^2} \mathbb{R}$, with $f \in C^\infty(P)$, and α is the projection $P \times_{f^2} \mathbb{R} \to \mathbb{R}$ with dilation $1/f = |\text{grad } \alpha|$. Since $|\text{grad } \alpha|$ is constant we conclude. $\square$

Remark 2.4. We point out that when the warping function is constant, then the warped product is actually a Riemannian product.

2.4. Isoparametric functions

Definition 2.5 (*[1]*). A smooth function $f : M \to \mathbb{R}$ is called *isoparametric* if locally, at points where $\operatorname{grad} f \neq 0$, there are smooth real functions γ and σ such that

$$\begin{cases} \text{(i) } |\operatorname{grad} f|^2 = \gamma \circ f, \\ \text{(ii) } \Delta f = \sigma \circ f. \end{cases} \tag{2.5}$$

Proposition 2.6. *A function f is isoparametric if and only if $\forall x \in M$ with $\operatorname{grad}_x f \neq 0$,*

$$\begin{cases} \text{(i) } \operatorname{grad}(|\operatorname{grad} f|) \textit{ is parallel to } \operatorname{grad} f, \\ \text{(ii) } \operatorname{grad}(\Delta f) \textit{ is parallel to } \operatorname{grad} f. \end{cases} \tag{2.6}$$

3. The biharmonicity of the inclusion

Let x_0 be an arbitrary point of M, and denote by

$$\begin{array}{rcl} \mathbf{i}_{x_0} : (N, h) & \to & (M \times_{f^2} N, G_{f^2}) \\ y & \to & (x_0, y) \end{array}$$

the inclusion map of N at the x_0 level in $M \times_{f^2} N$. The goal of this section is to characterize the biharmonicity of $\mathbf{i}_{x_0}$ in terms of f.

We note that the inclusion $\mathbf{i} : M \to M \times_{f^2} N$, defined by $\mathbf{i}(x) = (x, y_0)$, $y_0 \in N$, is always a totally geodesic map, thus harmonic for any warping function $f \in C^\infty(M)$.

We have

Theorem 3.1. *The bitension field of the inclusion $\mathbf{i}_{x_0} : (N, h) \to M \times_{f^2} N$ is given by*

$$\tau_2(\mathbf{i}_{x_0}) = \frac{n^2}{8}(\operatorname{grad}(|\operatorname{grad} f^2|^2), 0) \circ \mathbf{i}_{x_0}.$$

Proof. Let $\{F_a\}_{a=1}^n$ be a local orthonormal frame field on (N, h). By making use of Eq. (2.3) we obtain the following expression of the tension field of $\mathbf{i}_{x_0}$,

$$\begin{aligned} \tau(\mathbf{i}_{x_0}) &= \operatorname{trace}_h \nabla d\mathbf{i}_{x_0} \\ &= \sum_{a=1}^n \left\{ \nabla_{F_a} d\mathbf{i}_{x_0}(F_a) - d\mathbf{i}_{x_0}(\nabla^N_{F_a} F_a) \right\} \\ &= \sum_{a=1}^n \left\{ \widetilde{\nabla}_{(0,F_a)}(0, F_a) - (0, \nabla^N_{F_a} F_a) \right\} \circ \mathbf{i}_{x_0} \\ &= -\frac{n}{2}(\operatorname{grad} f^2, 0) \circ \mathbf{i}_{x_0}. \end{aligned}$$

Note that $\mathbf{i}_{x_0}$ is harmonic if and only if $(\operatorname{grad} f^2)_{x_0} = 0$.

Let us now compute the bitension field of the inclusion. First, using (2.2), we write down the rough Laplacian. We have

$$\begin{aligned} \nabla_{F_a} \tau(\mathbf{i}_{x_0}) &= -\frac{n}{2} \nabla_{F_a}(\operatorname{grad} f^2, 0) \circ \mathbf{i}_{x_0} \\ &= -\frac{n}{2} \left(\widetilde{\nabla}_{(0,F_a)}(\operatorname{grad} f^2, 0) \right) \circ \mathbf{i}_{x_0} \\ &= \left(-\frac{n}{4f^2} |\operatorname{grad} f^2|^2 (0, F_a) \right) \circ \mathbf{i}_{x_0}. \end{aligned}$$

Thus

$$\nabla_{F_a}\nabla_{F_a}\tau(\mathbf{i}_{x_0}) = -\frac{n}{2}\left\{\frac{1}{2f^2}|\mathrm{grad}\ f^2|^2\left((0, \nabla^N_{F_a}F_a) - \frac{1}{2}(\mathrm{grad}\ f^2, 0)\right)\right\}\circ \mathbf{i}_{x_0}.$$

Moreover,

$$\nabla_{\nabla^N_{F_a}F_a}\tau(\mathbf{i}_{x_0}) = -\frac{n}{2}\left(\frac{1}{2f^2}|\mathrm{grad}\ f^2|^2(0, \nabla^N_{F_a}F_a)\right)\circ \mathbf{i}_{x_0}$$

and by summing all terms we find

$$-\Delta\tau(\mathbf{i}_{x_0}) = \left(\frac{n^2}{8f^2}|\mathrm{grad}\ f^2|^2(\mathrm{grad}\ f^2, 0)\right)\circ \mathbf{i}_{x_0}.$$

Now, taking into account (2.4), we obtain

$$\begin{aligned}\mathrm{trace}_h\widetilde{R}(\mathrm{d}\mathbf{i}_{x_0}, \tau(\mathbf{i}_{x_0}))\mathrm{d}\mathbf{i}_{x_0} &= \frac{n^2}{4}\left\{-(\nabla^M_{\mathrm{grad}\ f^2}\mathrm{grad}\ f^2, 0) + \frac{1}{2f^2}|\mathrm{grad}\ f^2|^2(\mathrm{grad}\ f^2, 0)\right\}\circ \mathbf{i}_{x_0},\\ &= \frac{n^2}{8}\left\{-\left(\mathrm{grad}\ (|\mathrm{grad}\ f^2|^2), 0\right) + \frac{1}{f^2}|\mathrm{grad}\ f^2|^2(\mathrm{grad}\ f^2, 0)\right\}\circ \mathbf{i}_{x_0},\end{aligned}$$

and we conclude. □

Corollary 3.2. *The inclusion map* $\mathbf{i}_{x_0} : N \to M \times_{f^2} N$ *is a proper biharmonic map if and only if* x_0 *is not a critical point for* f^2*, but it is a critical point for* $|\mathrm{grad}\ f^2|^2$.

Corollary 3.3. *Each and every inclusion* $\mathbf{i}_x : N \to M \times_{f^2} N$*,* $x \in M$*, is a proper biharmonic map if and only if* $\mathrm{grad}\ f^2$ *is a non-zero constant norm vector field.*

Corollary 3.4. *Let* (M, g) *be a Riemannian manifold with a positive non-trivial affine function* f^2 *and let* (N, h) *be an arbitrary Riemannian manifold. Then any inclusion* $\mathbf{i}_x : N \to M \times_{f^2} N$*,* $x \in M$*, is a proper biharmonic map.*

Remark 3.5. The existence of a function with constant norm gradient on a given Riemannian manifold (M, g) was studied in [29], where the author underlined that such existence is considerably controlled by the Ricci curvature of M. Of course, any affine function has constant norm gradient and, conversely, if M is a complete connected Riemannian manifold of non-negative Ricci curvature, then any smooth function with constant norm gradient is an affine function. We point out that functions with constant norm gradient form a subclass of *transnormal functions*, i.e. functions which satisfy the first of the isoparametric conditions (2.5).

An important example of Riemannian manifold admitting a non-affine function of constant norm gradient is provided by the warped products: the projection $\overline{\pi} : \mathbb{R} \times_{f^2} N \to \mathbb{R}$ has constant norm gradient. Another such example is given by Busemann functions on Hadamard manifolds. Also, in [29] the author offers a wide range of examples of functions with $|\mathrm{grad}\ \alpha| = 1$, other than affine functions, on the m-dimensional hyperbolic space and gives a characterization of the Busemann functions from this point of view.

3.1. Examples

3.1.1. The compact case

Take M to be compact and let $f \in C^\infty(M)$ be a non-constant positive function. Then there exists $x_0 \in M$ a maximum point for $|\mathrm{grad}\ f^2|^2$, so the inclusion $\mathbf{i}_{x_0} : N \to M \times_{f^2} N$ is a proper biharmonic map.

Note that the equivalent conditions of Corollary 3.3 imply the non-existence of critical points for f^2 on M. As a consequence, if M is compact the inclusions $\mathbf{i}_x : N \to M \times_{f^2} N$, $x \in M$ are all biharmonic if and only if they are harmonic, and this holds only when f is constant.

3.1.2

We shall now apply Theorem 3.1 in order to obtain new examples of biharmonic maps. Take M to be

$$\mathbb{R}^m_+ = \{(x^1, \ldots, x^m) \in \mathbb{R}^m : x^i > 0, \forall i = 1, \ldots, m\}$$

and let (N, h) be arbitrary. By Corollary 3.3 and Remark 3.5 all the inclusions $\mathbf{i}_x : N \to \mathbb{R}^m_+ \times_{f^2} N$ are proper biharmonic maps if and only if there exists $x_0 \in \mathbb{R}^m_+$ and $a \in \mathbb{R}_+$ such that $f^2(x) = \langle x_0, x\rangle + a$, $\forall x \in \mathbb{R}^m_+$. When $m = 1$ we obtain a generalization of rotational surfaces with all parallels proper biharmonic.

Another example is obtained by taking M as $\mathbb{R}^m \setminus \{0\}$ and $f(x) = \sqrt{|x|}$, for $x \in M$. In this case grad $f^2(x) = \frac{x}{|x|}$, and, as before, $|\text{grad } f^2|$ is constant and the hypotheses of Corollary 3.3 are satisfied. This can be also verified by noting that f^2 is in fact the projection onto the first factor of the warped product $(0, \infty) \times_{t^2} \mathbb{S}^{m-1} = \mathbb{R}^m \setminus \{0\}$ and using Remark 3.5. We underline the fact that f^2 is no longer, as in the previous examples, an affine function on M.

3.1.3. Biharmonic inclusions in spheres as warped products

By using Theorem 3.1 we find a well known example of biharmonic map. Let $\mathbb{S}^m$ be the m-dimensional unit Euclidean sphere. Then, for a given point $p \in \mathbb{S}^m$, the space $\mathbb{S}^m \setminus \{\pm p\}$ is the warped product

$$(0, \pi) \times_{\sin^2 t} \mathbb{S}^{m-1}.$$

Consider now the inclusion map

$$\mathbf{i}_{t_0} : \mathbb{S}^{m-1} \to (0, \pi) \times_{\sin^2 t} \mathbb{S}^{m-1}.$$

By Theorem 3.1, $\mathbf{i}_{t_0}$ is a proper biharmonic map if and only if $t_0 \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$. This example was first obtained in a different manner in the study of proper biharmonic submanifolds of $\mathbb{S}^m$ (see [9,10]).

3.1.4. Biharmonic inclusions in rotational surfaces in $\mathbb{R}^3$ as warped products

Let $S \subset \mathbb{R}^3$ be a rotational surface obtained by rotating the arc length parameterized curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$,

$$\alpha : \begin{cases} x = x(t) \\ y = 0 \\ z = z(t) \end{cases}$$

in the xz-plane, around the z-axis.

It is immediate to see that the rotational surface endowed with the induced metric can be expressed as the warped product

$$S = I \times_{x^2} \mathbb{S}^1.$$

By Corollary 3.2, the inclusion $\mathbf{i}_{t_0} : \mathbb{S}^1 \to S$ is proper biharmonic if and only if t_0 satisfies

$$\dot{x}^2(t_0) + x(t_0)\ddot{x}(t_0) = 0 \quad \text{and} \quad \dot{x}(t_0) \neq 0.$$

We recover in this way a result obtained in a different context in [11].

4. The case of product maps

In this section we give a method to construct proper biharmonic maps of product type.

Let $\mathbf{1}_M : M \to M$ be the identity map of M and $\psi : N \to N$ be an arbitrary harmonic map. The product map

$$\phi = \mathbf{1}_M \times \psi : M \times N \to M \times N$$

is clearly a harmonic map. If we modify the product metric on $M \times N$ (either as the domain or the codomain), using a warping function $f \in C^\infty(M)$, then the product map may fail to remain harmonic; nonetheless we can find the condition on f so that the product map preserves its biharmonicity.

4.1

We first examine the case when the domain is endowed with a warped metric, thus we are looking at the product map

$$\overline{\phi} = \overline{\mathbf{1}_M \times \psi} : M \times_{f^2} N \to M \times N. \tag{4.1}$$

We have

Theorem 4.1. *Let $\psi : N \to N$ be a harmonic map and let $f \in C^\infty(M)$ be a positive function. Then $\overline{\phi} = \overline{\mathbf{1}_M \times \psi} : M \times_{f^2} N \to M \times N$ is a proper biharmonic map if and only if f is a non-constant solution of*

$$\mathrm{trace}_g \nabla^2 \mathrm{grad}\ \ln f + \mathrm{Ricci}^M(\mathrm{grad}\ \ln f) + \frac{n}{2}\mathrm{grad}\ (|\mathrm{grad}\ \ln f|^2) = 0. \tag{4.2}$$

Proof. Let $\{E_i\}_{i=1}^m$ and $\{F_a\}_{a=1}^n$ be local orthonormal frame fields on (M, g) and (N, h), respectively. Then $\{(E_i, 0), \frac{1}{f}(0, F_a)\}$ is a local orthonormal frame field on $(M \times_{f^2} N, G_{f^2})$. Using the harmonicity of ψ, the tension field of $\overline{\phi}$ is

$$\begin{aligned}
\tau(\overline{\phi}) &= \mathrm{trace}_{G_{f^2}} \nabla \mathrm{d}\overline{\phi} \\
&= \sum_{i=1}^m \left\{\nabla_{(E_i,0)} \mathrm{d}\overline{\phi}(E_i, 0) - \mathrm{d}\overline{\phi}(\widetilde{\nabla}_{(E_i,0)}(E_i, 0))\right\} + \frac{1}{f^2}\sum_{a=1}^n \left\{\nabla_{(0,F_a)} \mathrm{d}\overline{\phi}(0, F_a) - \mathrm{d}\overline{\phi}(\widetilde{\nabla}_{(0,F_a)}(0, F_a))\right\} \\
&= \frac{1}{f^2}(0, \tau(\psi)) + \frac{1}{f^2}\sum_{a=1}^n \mathrm{d}\overline{\phi}\left(\frac{1}{2}h(F_a, F_a)(\mathrm{grad}\ f^2, 0)\right) \\
&= n\,(\mathrm{grad}\ \ln f, 0)\,.
\end{aligned}$$

From the expression of the curvature tensor field for the Riemannian product $M \times N$ we get

$$\begin{aligned}
\mathrm{trace}_{G_{f^2}} R\left(\mathrm{d}\overline{\phi}, \tau(\overline{\phi})\right)\mathrm{d}\overline{\phi} &= n\left\{\sum_{i=1}^m R\left((E_i, 0), (\mathrm{grad}\ \ln f, 0)\right)(E_i, 0)\right. \\
&\qquad \left. + \frac{1}{f^2}\sum_{a=1}^m R\left((0, \mathrm{d}\psi(F_a)), (\mathrm{grad}\ \ln f, 0)\right)(0, \mathrm{d}\psi(F_a))\right\} \\
&= -n\left(\mathrm{Ricci}^M(\mathrm{grad}\ \ln f), 0\right).
\end{aligned}$$

Also, as $\nabla_{(0,X_2)}(Y_1, 0) = 0$ for all $X_1 \in C(TM)$ and $Y_2 \in C(TN)$, we get $\nabla_{(0,X_2)}\tau(\overline{\phi}) = 0$, thus

$$\begin{aligned}
-\Delta\tau(\overline{\phi}) &= \mathrm{trace}_{G_{f^2}} \nabla^2\tau(\overline{\phi}) \\
&= \sum_{i=1}^m \left\{\nabla_{(E_i,0)}\nabla_{(E_i,0)}\tau(\overline{\phi}) - \nabla_{\widetilde{\nabla}_{(E_i,0)}(E_i,0)}\tau(\overline{\phi})\right\} + \frac{1}{f^2}\sum_{a=1}^n \left\{\nabla_{(0,F_a)}\nabla_{(0,F_a)}\tau(\overline{\phi}) - \nabla_{\widetilde{\nabla}_{(0,F_a)}(0,F_a)}\tau(\overline{\phi})\right\} \\
&= n\left(\mathrm{trace}_g\, \nabla^2 \mathrm{grad}\ \ln f + n\nabla_{\mathrm{grad}\ \ln f}\mathrm{grad}\ \ln f, 0\right).
\end{aligned}$$

Finally, summing the two terms up, we find

$$\tau_2(\overline{\phi}) = n\left(\mathrm{trace}_g \nabla^2 \mathrm{grad}\ \ln f + \mathrm{Ricci}^M(\mathrm{grad}\ \ln f) + \frac{n}{2}\mathrm{grad}\ (|\mathrm{grad}\ \ln f|^2), 0\right),$$

and we conclude. □

Remark 4.2. We note that, besides its harmonicity, the map ψ gives no other contribution to the bitension field of $\overline{\phi}$. This enables us to construct a wide range of examples of proper biharmonic maps of product type (4.1).

Definition 4.3. A function $f \in C^\infty(M)$ is called a *biharmonic warping function* if the identity $\overline{\mathbf{1}} : M \times_{f^2} N \to M \times N$ is a proper biharmonic map for some N of dimension greater than zero.

From Theorem 4.1 we immediately deduce

Proposition 4.4. *Let $\psi : N \to N$ be a harmonic map, and $f \in C^\infty(M)$ a positive non-constant function. Then $\mathbf{1}_M \times \psi : M \times_{f^2} N \to M \times N$ is a proper biharmonic map if and only if f is a biharmonic warping function.*

Corollary 4.5. *Let M be non-compact and $f \in C^\infty(M)$ a positive non-constant function such that* $\ln f$ *is an affine function on M. Then f is a biharmonic warping function.*

Proof. Since $\ln f$ is an affine function then grad $\ln f$ is a Killing vector field of constant norm. Consequently (4.2) is satisfied. □

Remark 4.6. Eq. (4.2) can be written in the form

$$-\Delta_H \operatorname{grad} \ln f + 2\mathrm{Ricci}^M(\operatorname{grad} \ln f) + \frac{n}{2}\operatorname{grad}(|\operatorname{grad} \ln f|^2) = 0, \tag{4.3}$$

where $\Delta_H : C(TM) \to C(TM)$, is defined by $\Delta_H X = -\mathrm{trace}_g \nabla^2 X + \mathrm{Ricci}^M(X)$.

Using the musical isomorphisms $\sharp : T_x^*M \to T_xM$, $\flat : T_xN \to T_x^*M$, $(x \in M)$, the condition (4.3) can be rewritten as

$$-\Delta\omega + \frac{n}{2}\mathrm{d}|\omega|^2 + 2(\mathrm{Ricci}^M \omega^\sharp)^\flat = 0, \tag{4.4}$$

where $\omega = \mathrm{d}(\ln f)$.

Proposition 4.7. *Let M be a compact Riemannian manifold of negative Ricci curvature. Then there exist no biharmonic warping functions on M.*

Proof. Eq. (4.4) together with the Weitzenböck formula for 1-forms on M (see, for example, [14]),

$$\frac{1}{2}\Delta|\omega|^2 = \langle \Delta\omega, \omega\rangle - |\nabla\omega|^2 - \mathrm{Ricci}^M(\omega^\sharp, \omega^\sharp),$$

implies

$$\frac{1}{2}\Delta|\omega|^2 = \frac{n}{2}\langle \mathrm{d}|\omega|^2, \omega\rangle - |\nabla\omega|^2 + \mathrm{Ricci}^M(\omega^\sharp, \omega^\sharp).$$

Consider now x_0 to be a maximum point of $|\omega|^2$. Then $(\mathrm{d}|\omega|^2)_{x_0} = 0$ and $(\Delta|\omega|^2)_{x_0} \geq 0$, and, as Ricci^M is negative definite, we conclude. □

Eq. (4.4) also assumes a simpler form when M is an Einstein manifold and we have

Proposition 4.8. *Let M be an Einstein space. If $\rho \in C^\infty(M)$ is an isoparametric function then, away from critical points, it admits a local reparameterization f which is a biharmonic warping function.*

Proof. Consider c to be the Einstein constant of M. By the hypothesis, $|\mathrm{d}\rho|^2 = \gamma \circ \rho$ and, by choosing s to be a solution of the differential equation $s' = 1/\sqrt{\gamma}$, we can reparameterize ρ such that $s = s \circ \rho$ has $|ds|^2 = 1$. The function s is isoparametric with $\Delta s = \sigma \circ s$, for a smooth real function σ. We are looking for $\alpha = \alpha \circ s$, such that $\omega = \mathrm{d}\alpha$ satisfies (4.4).

For the searched reparameterization α we have

$$\mathrm{d}\alpha = (\alpha' \circ s)\mathrm{d}s,$$
$$\Delta\alpha = (-\alpha'' + \alpha'\sigma) \circ s,$$

and from here

$$\Delta\omega = \mathrm{d}\mathrm{d}^*\mathrm{d}\alpha = \mathrm{d}(\Delta\alpha) = (-\alpha''' + \alpha''\sigma + \alpha'\sigma') \circ s\, \mathrm{d}s,$$
$$\mathrm{d}|\omega|^2 = 2(\alpha'\alpha'') \circ s\, \mathrm{d}s,$$
$$(\mathrm{Ricci}^M \omega^\sharp)^\flat = c\,\omega = c\alpha' \circ s\, \mathrm{d}s.$$

By substituting into (4.4), we obtain a second order ordinary differential equation in $y = \alpha'$

$$y'' - \sigma y' + nyy' + (2c - \sigma')y = 0. \tag{4.5}$$

The general theory of differential equations offers local solutions for this type of problem. Finally, $\alpha(t) = \int y(t)\mathrm{d}t$, t being the real parameter, and $f = e^{\alpha \circ s}$. □

Example 4.9. Let $M = \mathbb{R}^m \setminus \{0\}$, $m \neq 2$, be endowed with the canonical metric and $s : M \to \mathbb{R}$, $s(x^1, x^2, \ldots, x^m) = \sqrt{(x^1)^2 + \cdots + (x^m)^2}$. Then s is isoparametric with $|\mathrm{d}s|^2 = 1$ and $\Delta s = -(m-1)/s$, so $\sigma(t) = -(m-1)/t$. In this case (4.5) becomes

$$y'' + \frac{m-1}{t} y' + nyy' - \frac{m-1}{t^2} y = 0. \tag{4.6}$$

Looking for particular solutions of type $y = a/t$, $a \in \mathbb{R}$, for (4.6) we get $a = 0$ or $a = \frac{2(2-m)}{n}$, so $\alpha(t) = \frac{2(2-m)}{n} \ln t$ and $f = s^{\frac{2(2-m)}{n}}$.

Example 4.10. When $M = \mathbb{R}$ we can classify all the biharmonic warping functions. In fact, in this case, Eq. (4.2) becomes

$$\alpha''' + n\alpha'\alpha'' = 0, \tag{4.7}$$

where $\alpha = \ln f$. We are only interested in the global solutions of (4.7). Considering the case when $\alpha'' = 0$ we obtain the biharmonic warping functions generated by the affine functions on $\mathbb{R}$,

$$f(t) = e^{a_1 t + a_2}, \quad a_1, a_2 \in \mathbb{R}.$$

Assume now that $\alpha'' \neq 0$. The general (globally defined) solution of (4.7) is

$$\alpha(t) = \ln \left(a_1 \left(\cosh(a_2 t + a_3)\right)^{\frac{2}{n}} \right),$$

with $a_1, a_2 > 0$, and $a_3 \in \mathbb{R}$. This drives to

$$f(t) = a_1 \left(\cosh(a_2 t + a_3)\right)^{\frac{2}{n}}.$$

In addition to these we obtain the local solutions:

$$f(t) = a_1 \left(\sinh(a_2 t + a_3)\right)^{\frac{2}{n}},$$
$$f(t) = a_1 \left(\cos(a_2 t + a_3)\right)^{\frac{2}{n}},$$
$$f(t) = a_2 (t + a_3)^{\frac{2}{n}},$$

where $a_1, a_2 > 0$ and $a_3 \in \mathbb{R}$.

We shall now analyze the case of the projection $\overline{\pi} : M \times_{f^2} N \to M$. Recall that the projection onto the second factor $\overline{\eta} : M \times_{f^2} N \to N$ is a horizontally homothetic submersion with totally geodesic fibres, and thus harmonic (see, for example, [3, pag. 53]).

By computing the second fundamental form of $\overline{\pi}$ we get $\tau(\overline{\pi}) = n \text{ grad } (\ln f) \circ \overline{\pi}$, and the bitension field of $\overline{\pi}$ has the expression

$$\tau_2(\overline{\pi}) = n \left\{ \mathrm{trace}_g \nabla^2 \mathrm{grad}\ \ln f + \mathrm{Ricci}^M(\mathrm{grad}\ \ln f) + \frac{n}{2} \mathrm{grad}\ (|\mathrm{grad}\ \ln f|^2) \right\} \circ \overline{\pi}.$$

As a consequence,

Proposition 4.11. *The projection $\overline{\pi} : M \times_{f^2} N \to M$ is non-harmonic biharmonic if and only if f is a biharmonic warping function.*

4.2

Consider now the case of the product map

$$\widetilde{\phi} = \widetilde{\mathbf{1}_M \times \psi} : M \times N \to M \times_{f^2} N, \tag{4.8}$$

that is, when the product metric on the codomain is warped. In this case, as we shall see, the role of ψ increases, involving its energy density. The difficulties are due to the contribution of the curvature tensor field of $M \times_{f^2} N$ in the expression of the bitension field. We have

Theorem 4.12. *Let $\mathbf{1}_M : M \to M$ be the identity map, $\psi : N \to N$ be a harmonic map and let $f \in C^\infty(M)$ be a non-constant positive function. Then $\widetilde{\phi} = \widetilde{\mathbf{1}_M \times \psi} : M \times N \to M \times_{f^2} N$ is a proper biharmonic map if and only if*

$$e(\psi)\left(-\operatorname{trace}_g \nabla^2 \operatorname{grad} f^2 - \operatorname{Ricci}^M(\operatorname{grad} f^2) + \frac{e(\psi)}{2}\operatorname{grad}(|\operatorname{grad} f^2|^2)\right) + (\Delta e(\psi))\operatorname{grad} f^2 = 0 \tag{4.9}$$

and

$$\mathrm{d}\psi(\operatorname{grad} e(\psi)) = 0. \tag{4.10}$$

Remark 4.13. 1. Using the same argument of Corollary 4.5, if M is non-compact, $\psi : N \to N$ is a harmonic map with constant energy density, and $f \in C^\infty(M)$ is a non-constant function such that f^2 is an affine function on M, then $\widetilde{\mathbf{1}_M \times \psi} : M \times N \to M \times_{f^2} N$ is a proper biharmonic map.

2. As in Proposition 4.7, if M is compact of negative Ricci curvature and $\psi : N \to N$ is a harmonic map of constant energy density, there exist no non-constant functions f that could render $\widetilde{\mathbf{1}_M \times \psi} : M \times N \to M \times_{f^2} N$ proper biharmonic.

3. Let M be an Einstein space and $\psi : N \to N$ a harmonic map of constant energy density λ. The analogous of (4.4) is

$$-\Delta\omega - \frac{\lambda}{2}\mathrm{d}|\omega|^2 + 2(\operatorname{Ricci}^M \omega^\sharp)^\flat = 0, \tag{4.11}$$

where $\omega = \mathrm{d}(f^2)$. Using the same argument as in Proposition 4.8 we conclude that if $\rho \in C^\infty(M)$ is an isoparametric function, then it admits, away from critical points, a local reparameterization $f : U \subset M \to (0, \infty)$ such that $\widetilde{\mathbf{1}_U \times \psi} : U \times N \to U \times_{f^2} N$ is a proper biharmonic map.

Example 4.14. If $M = \mathbb{R}$ and ψ the identity, then Eq. (4.9) becomes

$$\alpha''' - \frac{n}{2}\alpha'\alpha'' = 0,$$

where $\alpha = f^2$. In this case we have no global solutions.

When $\alpha'' = 0$ we get the local solutions, provided by affine functions on $(0, \infty)$,

$$f(t) = \sqrt{a_1 t + a_2}, \quad a_1 > 0, a_2 \geq 0.$$

For $\alpha'' \neq 0$ we obtain the local solutions:

$$f(t) = \sqrt{a_3 - \ln(\sinh(a_1 t + a_2))^{\frac{4}{n}}},$$
$$f(t) = \sqrt{a_3 - \ln(\cosh(a_1 t + a_2))^{\frac{4}{n}}},$$
$$f(t) = \sqrt{a_3 - \ln(\cos(a_1 t + a_2))^{\frac{4}{n}}},$$
$$f(t) = \sqrt{a_3 - \ln(t + a_2)^{\frac{4}{n}}},$$

where $a_1 > 0$.

Thus, we have the full classification of warping functions that render the identity $\widetilde{\mathbf{1}}$ a proper biharmonic map.

5. Axially symmetric biharmonic maps

This section is devoted to the study of the biharmonicity of axially symmetric maps by using the warped product setting.

Definition 5.1. A map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R} \times_{f^2} \mathbb{S}^{n-1}$ is called *axially symmetric* if there exist a map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ and a function $\rho : (0, \infty) \to \mathbb{R}$ such that

$$\phi = \rho \times \varphi : (0, \infty) \times_{t^2} \mathbb{S}^{m-1} \to \mathbb{R} \times_{f^2} \mathbb{S}^{n-1}.$$

We first determine the tension field of an axially symmetric map ϕ. Take $\partial/\partial t \in C(T(0, \infty))$, $\partial/\partial s \in C(T(0, \infty))$ and let $\{E_i\}_{i=1}^{m-1}$ be a local orthonormal frame field on the unit Euclidean sphere $\mathbb{S}^{m-1}$. Then

$$\begin{aligned} \nabla \mathrm{d}\phi\left((\partial/\partial t, 0), (\partial/\partial t, 0)\right) &= \left(\nabla_{\partial/\partial t} \mathrm{d}\rho(\partial/\partial t) - \mathrm{d}\rho\left(\nabla_{\partial/\partial t}\partial/\partial t\right), 0\right) \\ &= \rho''\,(\partial/\partial s, 0) \circ \phi \end{aligned}$$

and

$$\begin{aligned} \nabla \mathrm{d}\phi\left((0, E_i), (0, E_i)\right) &= \nabla_{(0,E_i)}(0, \mathrm{d}\varphi(E_i)) - \mathrm{d}\phi\left(\widetilde{\nabla}_{(0,E_i)}(0, E_i)\right) \\ &= \left(0, \nabla_{E_i}\mathrm{d}\varphi(E_i)\right) - \frac{1}{2}\langle \mathrm{d}\varphi(E_i), \mathrm{d}\varphi(E_i)\rangle(\mathrm{grad}\ f^2, 0) \circ \phi \\ &\quad - \mathrm{d}\phi\left((0, \nabla_{E_i} E_i) - \frac{1}{2}(\mathrm{grad}\ t^2, 0)\right) \\ &= \left(t\rho' - \langle \mathrm{d}\varphi(E_i), \mathrm{d}\varphi(E_i)\rangle\,(ff') \circ \rho\right)(\partial/\partial s, 0) \circ \phi + (0, \nabla \mathrm{d}\varphi(E_i, E_i)). \end{aligned}$$

As $\{(\partial/\partial t, 0), \frac{1}{t}(0, E_i)\}_{i=1}^{m-1}$ is a local orthonormal frame field on $(0, \infty) \times_{t^2} \mathbb{S}^{m-1}$, by summing all of the above, we get

$$\tau(\phi)_{(t,x)} = F(t, x)\left(\frac{\partial}{\partial s}, 0\right)_{\phi(t,x)} + \frac{1}{t^2}(0, \tau(\varphi)_x), \tag{5.1}$$

where $F : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ is defined by

$$F(t, x) = \rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{2e(\varphi)(x)}{t^2} f(\rho(t)) f'(\rho(t)). \tag{5.2}$$

We recall that a map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ is an eigenmap if it is harmonic and of constant energy density (see, for example, [3, pag. 77]. Then, analyzing expression (5.1) of the tension field of ϕ we have

Proposition 5.2. *An axially symmetric map $\phi = \rho \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R} \times_{f^2} \mathbb{S}^{n-1}$ is harmonic if and only if either φ is an eigenmap of eigenvalue $2e(\varphi) = 2k \geq 0$ and ρ is a non-constant solution of*

$$\rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{2k}{t^2} f(\rho(t)) f'(\rho(t)) = 0, \tag{5.3}$$

or φ is harmonic, $\rho = \rho_0$ is constant and $f(\rho_0) f'(\rho_0) = 0$.

Remark 5.3. A general equation which includes (5.3) as a special case was studied in [1].

Consider now φ to be a harmonic map and let us compute the rough Laplacian. We have

$$\nabla_{(\partial/\partial t, 0)}\tau(\phi) = \frac{\partial F}{\partial t}\ (\partial/\partial s, 0) \circ \phi,$$

so

$$\nabla^2 \tau(\phi)\left((\partial/\partial t, 0), (\partial/\partial t, 0)\right) = \frac{\partial^2 F}{\partial t^2}\ (\partial/\partial s, 0) \circ \phi.$$

Also,

$$\begin{aligned}\nabla_{(0,E_i)}\tau(\phi) &= E_i(F)\,(\partial/\partial s,0)\circ\phi + F\nabla_{(0,E_i)}\,(\partial/\partial s,0)\circ\phi\\ &= E_i(F)\,(\partial/\partial s,0)\circ\phi + \frac{f'(\rho)}{f(\rho)}F\,(0,\mathrm{d}\varphi(E_i))\,,\end{aligned}$$

so

$$\begin{aligned}\nabla_{(0,E_i)}\nabla_{(0,E_i)}\tau(\phi) = {}&\Big(E_i(E_i(F)) - Ff'^2(\rho)\langle\mathrm{d}\varphi(E_i),\mathrm{d}\varphi(E_i)\rangle\Big)\,(\partial/\partial s,0)\circ\phi\\ &+2\frac{f'(\rho)}{f(\rho)}E_i(F)\,(0,\mathrm{d}\varphi(E_i)) + \frac{f'(\rho)}{f(\rho)}F\,\big(0,\nabla_{E_i}\mathrm{d}\varphi(E_i)\big)\end{aligned}$$

and

$$\nabla_{\widetilde{\nabla}_{(0,E_i)}(0,E_i)}\tau(\phi) = \left(\big(\nabla_{E_i}E_i\big)(F) - t\frac{\partial F}{\partial t}\right)(\partial/\partial s,0)\circ\phi + F\frac{f'(\rho)}{f(\rho)}\big(0,\mathrm{d}\varphi(\nabla_{E_i}E_i)\big)\,.$$

From here, as $\tau(\varphi)=0$ and

$$\mathrm{trace}_{G_{t^2}}\widetilde{R}(\mathrm{d}\phi,\tau(\phi))\mathrm{d}\phi = 2F\frac{f(\rho)f''(\rho)}{t^2}e(\varphi)\,(\partial/\partial s,0)\circ\phi,$$

the bitension field of ϕ can be written as

$$\begin{aligned}\tau_2(\phi) = {}&\left\{\frac{\partial^2F}{\partial t^2} + \frac{m-1}{t}\frac{\partial F}{\partial t} - \frac{2e(\varphi)}{t^2}\Big(f'^2(\rho)+f(\rho)f''(\rho)\Big)F\right.\\ &+\left.\frac{2f(\rho)f'(\rho)}{t^4}\Delta e(\varphi)\right\}(\partial/\partial s,0)\circ\phi - \frac{4}{t^4}f'^2(\rho)\,(0,\mathrm{d}\varphi(\mathrm{grad}\,e(\varphi)))\,. \qquad (5.4)\end{aligned}$$

The calculations above lead to the following

Theorem 5.4. *Let $\varphi:\mathbb{S}^{m-1}\to\mathbb{S}^{n-1}$ be an eigenmap of eigenvalue $2k\geq 0$. Then $\phi=\rho\times\varphi:\mathbb{R}^m\backslash\{0\}\to\mathbb{R}\times_{f^2}\mathbb{S}^{n-1}$ is biharmonic if and only if ρ is a solution of*

$$F'' + \frac{m-1}{t}F' - \frac{2k}{t^2}\Big(f'^2(\rho)+f(\rho)f''(\rho)\Big)F = 0,$$

where

$$F = \rho'' + \frac{m-1}{t}\rho' - \frac{2k}{t^2}f(\rho)f'(\rho).$$

In the following we will treat the case when $k=0$, i.e. φ is constant.

Proposition 5.5. *The axially symmetric map $\phi=\rho\times\varphi_0:\mathbb{R}^m\setminus\{0\}\to\mathbb{R}\times_{f^2}\mathbb{S}^{n-1}$ is:*

(a) *harmonic if and only if*

$$\rho(t) = \begin{cases} c_1\ln t + c_2, & \text{when } m=2\\ c_1t^{2-m}+c_2, & \text{when } m\geq 3,\end{cases}$$

where $c_1, c_2\in\mathbb{R}$;

(b) *proper biharmonic if and only if*

$$\rho(t) = \begin{cases} c_1t^2\ln t + c_2t^2 + c_3\ln t + c_4, & \text{when } m=2\\ c_1\ln t + c_2t^2 + \dfrac{c_3}{t^2} + c_4, & \text{when } m=4\\ c_1t^{4-m} + c_2t^2 + c_3t^{2-m} + c_4, & \text{otherwise},\end{cases}$$

where $c_1^2+c_2^2\neq 0$.

Remark 5.6. Since the inclusion $\mathbb{R} \times \{\varphi_0\}$ in $\mathbb{R} \times_{f^2} \mathbb{S}^{n-1}$ is totally geodesic, the harmonicity of ϕ is equivalent to the harmonicity of ϕ thought of as a real function. In fact, in Proposition 5.5(a) we obtain the well-known formulae of harmonic radial functions on $\mathbb{R}^m \setminus \{0\}$. In the same way, using Proposition 4.4 in [28], for the biharmonic case, we get all the biharmonic radial functions on $\mathbb{R}^m \setminus \{0\}$.

5.1. Axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{R}^n \setminus \{0\}$

Since the Euclidean space $\mathbb{R}^m \setminus \{0\}$ can be expressed as the warped product

$$\mathbb{R}^m \setminus \{0\} = (0, \infty) \times_{t^2} \mathbb{S}^{m-1},$$

a smooth map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is *axially symmetric* if there exists a map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ and a function $\rho : (0, \infty) \to (0, \infty)$ such that, for $y \in \mathbb{R}^m \setminus \{0\}$,

$$\phi(y) = \rho(|y|)\varphi\left(\frac{y}{|y|}\right).$$

In this case the function $F : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ of (5.2) is given by

$$F(t, x) = \rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{2e(\varphi)(x)}{t^2}\rho(t),$$

so,

Proposition 5.7. *The axially symmetric map $\phi = \rho \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, with φ non-constant, is harmonic if and only if φ is an eigenmap of eigenvalue $2k > 0$ and*

$$\rho(t) = c_1 t^{A_1} + c_2 t^{A_2}, \tag{5.5}$$

where

$$A_{1,2} = \frac{-(m-2) \pm \sqrt{(m-2)^2 + 8k}}{2},$$

and $c_1, c_2 \geq 0$ with $c_1^2 + c_2^2 \neq 0$.

Proof. From Proposition 5.2, φ is an eigenmap of eigenvalue $2k > 0$ and ρ is a solution of

$$\rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{2k}{t^2}\rho(t) = 0. \tag{5.6}$$

By making the change of variable $t = e^x$, (5.6) becomes the following homogeneous linear equation with constant coefficients

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}x^2} + (m-2)\frac{\mathrm{d}\rho}{\mathrm{d}x} - 2k\rho = 0,$$

which has the desired solution. $\square$

Assume that φ is an eigenmap of eigenvalue $2k > 0$. Then the bitension field given in (5.4) becomes

$$\tau_2(\phi) = \left(F'' + \frac{m-1}{t}F' - \frac{2k}{t^2}F\right)\frac{\partial}{\partial s},$$

where

$$F(t) = \rho'' + \frac{m-1}{t}\rho' - \frac{2k}{t^2}\rho.$$

We shall now study the equation $\tau_2(\phi) = 0$.

Case I. $[\rho' = 0]$. Then $\tau_2(\phi) = 0$ is equivalent to $k = -m + 4$.

Recall that (see [15]) a map between spheres $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ is an eigenmap if and only if its components $\Phi^\alpha : \mathbb{S}^{m-1} \to \mathbb{R}$ are spherical harmonics of the same order, i.e. restrictions of harmonic homogeneous polynomial of the same degree on $\mathbb{R}^m$. Also, for any eigenmap $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ there exists $h \in \mathbb{N}$ (the homogeneity degree) such that its eigenvalue is

$$2e(\varphi) = 2k = h(m + h - 2).$$

Consequently,

1. for $m \geq 4$, ϕ cannot be biharmonic;
2. for $m = 3$, ϕ is a proper biharmonic map if and only if φ is an eigenmap of homogeneous degree $h = 1$; as an example, consider $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ to be the identity map;
3. for $m = 2$, ϕ is a proper biharmonic map if and only if φ is an eigenmap of homogeneous degree $h = 2$; the map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$, $\varphi(z) = \overline{z}^2$ provides us such an example.

Case II. $[\rho' \neq 0]$. In this case $\phi = \rho \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is a proper biharmonic map if and only if

$$\rho(t) = \begin{cases} c_1 t^3 + c_2 t \ln t + c_3 t + c_4 t^{-1}, & \text{when } m = 2 \text{ and } k = \dfrac{1}{2} \\ \dfrac{c_1}{2(m+2A_1)} t^{A_1+2} + \dfrac{c_2}{2(m+2A_2)} t^{A_2+2} + c_3 t^{A_1} + c_4 t^{A_2}, & \text{otherwise} \end{cases} \tag{5.7}$$

where $A_{1,2} = \frac{-(m-2)\pm\sqrt{(m-2)^2+8k}}{2}$, $c_1^2 + c_2^2 \neq 0$ and c_1, c_2, c_3, c_4 arbitrary such that ρ takes values in $(0, \infty)$.

Remark 5.8. An important class of axially symmetric diffeomorphisms of $\mathbb{R}^m \setminus \{0\}$ is given by

$$\begin{array}{rcl} \phi : \mathbb{R}^m \setminus \{0\} & \longrightarrow & \mathbb{R}^m \setminus \{0\} \\ y & \longmapsto & y/|y|^\ell \end{array}$$

($\ell \neq 0, 1$), which for $\ell = 2$ provides the well known Kelvin transformation. For these maps, $\rho(t) = 1/t^{\ell-1}$ and $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ is the identity map. An easy computation shows that ϕ is harmonic if and only if $m = \ell$.

Using (5.7) it follows that ϕ is a proper biharmonic map if and only if $m = \ell + 2$. For $\ell = 2$ this result was first obtained in [2].

We also note that the proper biharmonic map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$, $\phi(y) = y/|y|^{m-2}$, is harmonic with respect to the conformal metric on the domain given by $\widetilde{g} = |y|^{\frac{4}{3-m}} g_{\mathrm{can}}$. This property is similar to that of the Kelvin transformation proved by Fuglede in [16, pag. 143].

5.2. *Axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{S}^n \setminus \{\pm p\}$*

The space $\mathbb{S}^n \setminus \{\pm p\}$, where $p \in \mathbb{S}^m$, can be described as a warped product

$$\mathbb{S}^n \setminus \{\pm p\} = (0, \pi) \times_{\sin^2 s} \mathbb{S}^{n-1},$$

So, a smooth map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{S}^n \setminus \{\pm p\}$ is *axially symmetric* if there exists a map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ and a function $\rho : (0, \infty) \to (0, \pi)$ such that for $y \in \mathbb{R}^m \setminus \{0\}$,

$$\phi(y) = \left(\cos\rho(|y|), \varphi\left(\frac{y}{|y|}\right)\sin\rho(|y|)\right).$$

The map $F : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ of (5.2) is given in this case by

$$F(t, x) = \rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{1}{t^2}e(\varphi)\sin 2\rho(t).$$

Proposition 5.9. *The axially symmetric map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{S}^n \setminus \{\pm p\}$ is harmonic if and only if either φ is harmonic and $\rho = \frac{\pi}{2}$, or φ is an eigenmap of eigenvalue $2e(\varphi) = 2k \geq 0$ and ρ is a non-constant solution of*

$$\rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{k}{t^2}\sin 2\rho(t) = 0. \tag{5.8}$$

For the biharmonicity we will only study the case when φ is an eigenmap of eigenvalue $2k > 0$ and $\rho = \rho_0 \neq \frac{\pi}{2}$ is constant, i.e. the image of ϕ is contained in a hypersphere of $\mathbb{S}^n$. We underline here the fact that $e(\phi)$ is not constant, thus Theorem 1.7 in [23] does not apply. The function F becomes $F = F(t) = -\frac{k}{t^2}\sin 2\rho_0$, so the bitension field of ϕ assumes the form

$$\tau_2(\phi) = \frac{2k\sin 2\rho_0}{t^4}\,(m - 4 + k\cos 2\rho_0)\,\frac{\partial}{\partial s},$$

and we get

Proposition 5.10. *Let $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be an eigenmap of eigenvalue $2k > 0$. The axially symmetric map $\phi = \rho_0 \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{S}^n \setminus \{\pm p\}$ is proper biharmonic if and only if $\frac{4-m}{k} \in (-1, 1)$ and $\rho_0 \in \{\frac{1}{2}\arccos\frac{4-m}{k}, \pi - \frac{1}{2}\arccos\frac{4-m}{k}\}$.*

Corollary 5.11. *Let $\varphi : \mathbb{S}^3 \to \mathbb{S}^{n-1}$ be an arbitrary non-constant eigenmap. Then $\phi = \rho_0 \times \varphi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{S}^n \setminus \{\pm p\}$ is a proper biharmonic map if and only if $\rho_0 \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$.*

In the following we shall present some examples of eigenmaps satisfying the condition $\frac{4-m}{k} \in (-1, 1)$.

Example 5.12. The map $\varphi_h : \mathbb{S}^1 \to \mathbb{S}^1$, $\varphi_h(z) = \bar{z}^h$, $h > 2$, is an eigenmap with $\frac{4-m}{k} = \frac{4}{h^2} \in (0, 1)$.

Example 5.13. The Veronese map $\widetilde{\varphi}_{m-1} : \mathbb{S}^{m-1}(\sqrt{\frac{2m}{m-1}}) \to \mathbb{S}^{\frac{(m-1)(m+2)}{2}-1}$, $m \geq 3$, induces an eigenmap $\varphi_{m-1} : \mathbb{S}^{m-1} \to \mathbb{S}^{\frac{(m-1)(m+2)}{2}-1}$ with $\frac{4-m}{k} = \frac{4-m}{m} \in (-1, 1)$, $\forall m \geq 3$.

Example 5.14. The Hopf maps $\varphi : \mathbb{S}^3 \to \mathbb{S}^2$, $\varphi : \mathbb{S}^7 \to \mathbb{S}^4$, $\varphi : \mathbb{S}^{15} \to \mathbb{S}^8$ also satisfy the above condition.

5.3. Axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{H}^n \setminus \{p\}$

Consider $\mathbb{R}^{n+1}$ equipped with the standard Lorenzian inner product

$$\langle v, w\rangle_1 = v^1 w^1 + \cdots + v^n w^n - v^{n+1} w^{n+1}, \quad v, w \in \mathbb{R}^{n+1}.$$

Then the hyperboloid model of $\mathbb{H}^n$ is (the upper sheet of the hyperboloid)

$$\mathbb{H}^n = H^n_+ = \{(x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1)^2 + \cdots + (x^n)^2 - (x^{n+1})^2 = -1, x^{n+1} \geq 1\},$$

equipped with the Riemannian metric given by the restriction to H^n_+ of $\langle\,,\,\rangle_1$.

Let $p = (0, \ldots, 0, 1)$, then $\mathbb{H}^n \setminus \{p\}$ is diffeomorphic to $(0, \infty) \times \mathbb{S}^{n-1}$ by

$$(x^1, \ldots, x^n, x^{n+1}) \longrightarrow \left(s, \frac{(x^1, \ldots, x^n)}{\sinh s}\right),$$

where $x^{n+1} = \cosh s$. In these new coordinates, the Riemannian metric on $\mathbb{H}^n \setminus \{p\}$ has the expression

$$g = \mathrm{d}s^2 + \sinh^2 s\; g_{\mathbb{S}^{n-1}},$$

so $\mathbb{H}^n \setminus \{p\}$ can be thought of as the warped product

$$\mathbb{H}^n \setminus \{p\} = (0, \infty) \times_{\sinh^2 s} \mathbb{S}^{n-1}.$$

Let $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{H}^n \setminus \{p\}$ be a smooth map. The map ϕ is *axially symmetric* if there exists a map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ and a function $\rho : (0, \infty) \to (0, \infty)$ such that for $y \in \mathbb{R}^m \setminus \{0\}$,

$$\phi(y) = \left(\cosh\rho(|y|), \varphi\left(\frac{y}{|y|}\right)\sinh\rho(|y|)\right).$$

Proposition 5.15. *The axially symmetric map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{H}^n \setminus \{p\}$ is harmonic if and only if φ is an eigenmap of eigenvalue $2e(\varphi) = 2k \geq 0$ and ρ is a solution of*

$$\rho''(t) + \frac{n-1}{t}\rho'(t) - \frac{k}{t^2}\sinh 2\rho(t) = 0. \tag{5.9}$$

For the biharmonicity, as for the case of the sphere, we will only consider the case when $\rho = \rho_0$ is constant and $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ is an eigenmap of eigenvalue $2k > 0$. We have

$$F = F(t) = -\frac{k}{t^2}\sinh 2\rho_0.$$

The bitension field of ϕ assumes the form

$$\tau_2(\phi) = \frac{2k\sinh 2\rho_0}{t^4}\,(m - 4 + k\cosh 2\rho_0)\,\frac{\partial}{\partial s}.$$

Proposition 5.16. *Let $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be an eigenmap of eigenvalue $2k > 0$.*

(a) *If $m \geq 3$, then the axially symmetric map $\phi = \rho_0 \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{H}^n \setminus \{p\}$ cannot be biharmonic.*
(b) *If $m = 2$, then the axially symmetric map $\phi = \rho_0 \times \varphi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{H}^n \setminus \{p\}$ is proper biharmonic if and only if φ has the homogeneity degree $h = 1$ and $\rho_0 = \frac{1}{2}\ln(4 + \sqrt{15})$.*

Example 5.17. The map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$, $\varphi(z) = \bar{z}$, is an eigenmap of homogeneity degree $h = 1$ and so gives a proper biharmonic map from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{H}^2 \setminus \{p\}$.

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